Wedge Product and Associativity

Definition

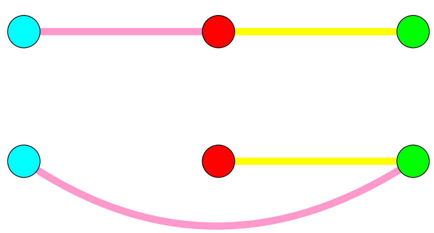
$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle$$

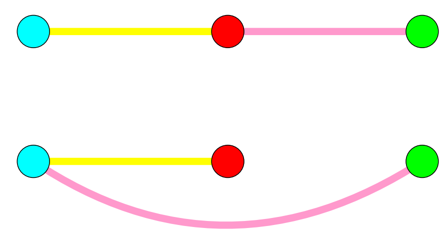
$$= \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \alpha \smile \beta(\tau(\sigma^{k+l}))$$

Non-associative

- Arises from difference in stencil.

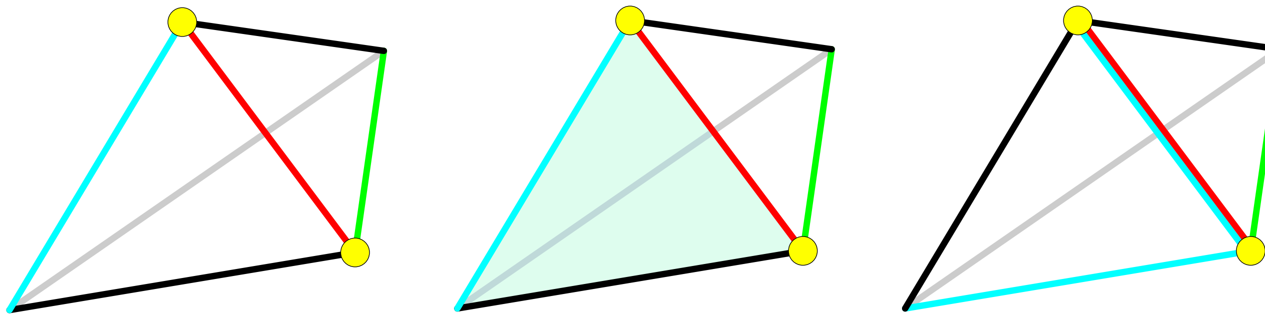
$\alpha \wedge (\beta \wedge \gamma)$	$(\alpha \wedge \beta) \wedge \gamma$
$\sum \alpha \smile (\sum \beta \smile \gamma)$	$(\sum \alpha \smile \beta) \smile \sum \gamma$





Associative for Closed Forms

- Since forms are closed, can rewrite sum so that the forms are all evaluated at a common vertex.



- Then, both sides can be written as,

$$\langle \alpha \wedge \beta \wedge \gamma, \sigma^{k+l+m} \rangle = \sum_{i=0}^{k+l+m} \sum_{\tau \in S^{k+l+m}} \alpha \smile \beta \smile \gamma$$

- Associativity important for showing that the algebraic definitions of contraction and Lie derivative are $\wedge$-antiderivations.

Discrete Differential Forms Revisited

Pointed Simplices

- A **Pointed Simplex** is a simplex σ^k with a distinguished vertex v_0.
- By fixing the base point v_0, the wedge product is naturally associative.

Candidate Basis

- To each pointed simplex (σ^k, v_0), we associate the following shape function,

$$\varphi_0 \mathbf{d}\varphi_1 \wedge \ldots \wedge \mathbf{d}\varphi_k.$$

- Need to construct finite dimensional function spaces that are closed under differential form operations.

Contraction

- **Algebraic**

$$\mathbf{i}_X \alpha = (-1)^{k(n-k)} * (*\alpha \wedge X^\flat) \,.$$

- **Dynamic**

$$\int_S i_X \beta = \left.\frac{d}{dt}\right|_{t=0} \int_{E_X^t(S)} \beta.$$

Lie Derivative

- **Algebraic**

$$\pounds_{\mathbf{X}}\omega = \mathbf{i}_{\mathbf{X}}\mathbf{d}\omega + \mathbf{d}\mathbf{i}_{\mathbf{X}}\omega\,.$$

- **Dynamic**

$$\int_S \pounds_X\beta = \left.\frac{d}{dt}\right|_{t=0}\int_{S_t}\beta\,.$$