A Discrete Theory of Connections on Principal Bundles

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Introduction

Motivation and Background

- Motivated by the desire to develop a theory of **Discrete Lagrangian Reduction**.
- **Discrete Connections** are necessary to represent the Discrete Lagrange-Poincaré operator, which is a coordinate representation of the reduced Euler-Lagrange operator that describes the reduced dynamics of a G-invariant discrete Lagrangian system.
- Constructed by considering a splitting of the **Discrete Atiyah sequence** of a principal bundle.
- Recovers continuous connections in a natural manner.
- Computational applications include Discrete Levi-Civita connections and curvatures for Discrete Riemannian manifolds.

Principal and Associated Bundles

Definitions

- A **Principal Bundle** is a manifold Q with a free *left* action $G \times Q \longrightarrow Q$ of a Lie group G, such that the natural projection $\pi : Q \longrightarrow Q/G$ is a submersion.
- An **Associated Bundle** $\tilde{M}$ with fibre M by definition,

$$\tilde{M} = Q \times_G M = (Q \times M)/G.$$

In particular, two associated bundles arise when considering the continuous and discrete Atiyah sequence of a principal bundle:

- $\tilde{\mathfrak{g}}$, where the action of G on $Q \times \mathfrak{g}$ is given by $g(q, \xi) = (gq, Ad_g \xi)$, and $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \longrightarrow Q/G$ is given by $\pi_{\mathfrak{g}}([q, \xi]_G) = \pi(q)$.
- $\tilde{G}$, where the action of G on $Q \times G$ is given by $g(q, h) = (gq, ghg^{-1})$, and $\pi_G : \tilde{G} \longrightarrow Q/G$ is given by $\pi_G([q, g]_G) = \pi(q)$.

Discrete Atiyah Sequence

Discrete Atiyah Sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{G} & \underset{(\pi_1, \mathcal{A}_d)}{\overset{i}{\rightleftarrows}} & (Q \times Q)/G & \underset{(\cdot,\cdot)^h}{\overset{(\pi,\pi)}{\rightleftarrows}} & S \times S & \longrightarrow & 0 \\
 & & \Big\| 1_{\tilde{G}} & & \Big\downarrow \alpha_{\mathcal{A}_d} & & \Big\| 1_{S \times S} & & \\
0 & \longrightarrow & \tilde{G} & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & \tilde{G} \oplus (S \times S) & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & S \times S & \longrightarrow & 0
\end{array}
$$

Maps

- $i : \tilde{G} \to (Q \times Q)/G$, where,

$$i([q, g]_G) = [q, gq]_G.$$

- $(\pi, \pi) : (Q \times Q)/G \to S \times S$, where,

$$(\pi, \pi)([q_0, q_1]_G) = (\pi q_0, \pi q_1).$$

Equivalent Representations of a Discrete Connection

Maps on the un-quotiented sequence

- Discrete Connection Form $\mathcal{A}_d : Q \times Q \to G$.
- Discrete Horizontal Lift $(\cdot, \cdot)_q^h : S \times S \to Q \times Q$.

Maps defining a splitting

- $(\pi_1, \mathcal{A}_d) : (Q \times Q)/G \to \tilde{G}$,
  related to the discrete connection form.
- $(\cdot, \cdot)^h : S \times S \to (Q \times Q)/G$,
  related to the discrete horizontal lift.

Relating the two sets

- The maps on the un-quotiented sequence induce splittings.
- The splittings extend by equivariance to recover the un-quotiented maps.

Discrete Connection Form

Definition

A **Discrete Connection Form** is a continuous map,

$$\mathcal{A}_d : Q \times Q \rightarrow G,$$

such that,

- $\mathcal{A}_d$ is G-equivariant.

$$\mathcal{A}_d \circ L_g = I_g \circ \mathcal{A}_d.$$

  This is the discrete analogue of the statement, $\mathcal{A} \circ L_g = Ad_g \circ \mathcal{A}$.

- $\mathcal{A}_d$ induces a splitting of the Discrete Atiyah sequence.

$$\mathcal{A}_d(i_q(g)) = g.$$

  This is the discrete analogue of the statement, $\mathcal{A}(\xi_Q) = \xi$.

Example

Discrete Mechanical Connection

- Given the point $(q_0, q_1) \in Q \times Q$, we construct the geodesic path $q_{01} : [0,1] \to Q$ with respect to the kinetic energy metric, such that $q_{01}(0) = q_0$, and $q_{01}(1) = q_1$.
- Project the geodesic path to the shape space, $x_{01}(t) \equiv \pi q_{01}(t)$, to obtain the curve x_{01} on S.
- Taking the horizontal lift of x_{01} to Q using the connection $\mathcal{A}$ yields $\tilde{q}_{01}$.
- There is a unique $g \in G$ such that $q_{01}(1) = g \cdot \tilde{q}_{01}(1)$.
- Define $\mathcal{A}_d(q_0, q_1) = g$.

Discrete Horizontal and Vertical Spaces

Horizontal Space

$$\mathrm{Hor}_q = \{(q, q') \in Q \times Q \mid \mathcal{A}_d(q, q') = e\}.$$

This is the discrete analogue of the statement $\mathrm{Hor}_q = \{v_q \in TQ \mid \mathcal{A}(v_q) = 0\}$.

Vertical Space

$$\mathrm{Ver}_q = \{(q, q') \in Q \times Q \mid (\pi, \pi)(q, q') = e_{S\times S}\} = \{i_q(g) \mid g \in G\}.$$

This is the discrete analogue of the statement $\mathrm{Ver}_q = \{v_q \in TQ \mid \pi_*(v_q) = 0\} = \{\xi_Q \mid \xi \in \mathfrak{g}\}$.

Horizontal Component

$$\mathrm{hor}(q_0, q_1) = (\cdot, \cdot)^h_{q_0} \circ (\pi, \pi)(q_0, q_1) = (q_0, (\mathcal{A}_d(q_0, q_1))^{-1} q_1).$$

Vertical Component

$$\mathrm{ver}(q_0, q_1) = i_{q_0} \circ (\pi_1, \mathcal{A}_d)(q_0, q_1) = (q_0, \mathcal{A}_d(q_0, q_1) q_0).$$

Decomposition

$$\mathrm{hor}(q_0, q_1) \cdot \mathrm{ver}(q_0, q_1) = (q_0, \mathcal{A}_d(q_0, q_1)(\mathcal{A}_d(q_0, q_1))^{-1} q_1) = (q_0, q_1).$$

Isomorphism between $(Q \times Q)/G$ and $(S \times S) \oplus \tilde{G}$

Discrete Atiyah Sequence

$$0 \longrightarrow \tilde{G} \underset{(\pi_1, \mathcal{A}_d)}{\overset{i}{\rightleftarrows}} (Q \times Q)/G \underset{(\cdot,\cdot)^h}{\overset{(\pi,\pi)}{\rightleftarrows}} S \times S \longrightarrow 0$$

Lemma

The map $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \to (S \times S) \oplus \tilde{G}$ defined by,

$$\alpha_{\mathcal{A}_d}([q_0, q_1]_G) = (\pi q_0, \pi q_1) \oplus [q_0, \mathcal{A}_d(q_0, q_1)]_G,$$

is a well-defined bundle isomorphism.

The inverse of α_{A_d} is given by,

$$\alpha^{-1}_{\mathcal{A}_d}((x_0, x_1) \oplus [q, g]_G) = [(e, g) \cdot (x_0, x_1)^h_q]_G,$$

for any $q \in Q$ such that $\pi q = x_0$.

Continuous Connections from Discrete Connections

- Given a discrete G-valued connection 1-form $\mathcal{A}_d : Q \times Q \longrightarrow G$, we associate with it a continuous $\mathfrak{g}$-valued connection 1-form $\mathcal{A} : TQ \longrightarrow \mathfrak{g}$ by the following construction,
$$\mathcal{A}([q(\cdot)]) = [\mathcal{A}_d(q(0), q(\cdot))],$$
where $[\cdot]$ denotes the equivalence class of curves associated with a tangent vector.

- More explicitly, given $v_q \in TQ$, we consider an associated curve $q : [0,1] \longrightarrow Q$, such that $v_q = d/dt|_{t=0}\, q(t)$, and construct the curve $g : [0,1] \longrightarrow G$, given by,
$$g(t) = \mathcal{A}_d(q(0), q(t)).$$
Then,
$$\mathcal{A}(v_q) = \left.\frac{d}{dt}\right|_{t=0} g(t).$$

Applications to Discrete Lagrangian Reduction

■ Discrete Lagrange–Poincaré operator

If in addition to the principal bundle structure, we have a discrete principal connection as described in the previous section, we can identify

$$Q^3/G \quad \text{with} \quad (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}).$$

Furthermore, each discrete G-valued connection 1-form $\mathcal{A}_d : Q \times Q \to G$ induces in the infinitesimal limit a continuous $\mathfrak{g}$-valued connection 1-form $\mathcal{A} : TQ \to \mathfrak{g}$. This continuous principal connection allows us to identify

$$T^*Q/G \quad \text{with} \quad T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

The discrete Lagrange–Poincaré operator $\mathcal{LP}_d(l_d) : (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}) \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. is derived from the reduced discrete Euler–Lagrange operator by making the appropriate identifications.

Conclusion

Summary

- Constructed a Discrete Atiyah sequence.
- Described equivalent representations of a discrete connection.
- Continuous connections as limits of discrete connections.
- Applications to Discrete Lagrangian Reduction.

Future Directions

- Further examine the links with groupoid theory.
- Computationally efficient methods of constructing discrete connections that limit to a prescribed continuous connection.
- Discrete Connections for Semidiscrete Principal Bundles.